Special Functions in OpenMath

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As usual, all errors are mine.

1 Preliminaries

The following principles were endorsed at the OpenMath Meeting at ISSAC 2001 and at the OpenMath Network Meeting in Berlin (August 2001).

- Many CDs, rather than a single "special functions CD".
- Short names, such as J, similar to mathematical notation.
- Use currying where appropriate, so that we can talk about J_ν rather than having to talk about $\lambda z : \mathrm{J}(\nu, z)$.
- As in the work on elementary functions, to stick as closely as possible to Abramovitz and Stegun, and to be as precise as possible.

1.1 Difficulties with these

- Just how many CDs? A CD for Laguerre, with one symbol in, or a CD for all families of orthogonal polynomials, or what?
  I have probably used too many CDs.
- Just what names are appropriate?
- Where is currying appropriate?
  Please tell me where you think I've got this wrong.
- How much should we follow A+S: symbols defined in footnotes or as alternates? Do we need erfc? Every last *positive* wrinkle??
- A+S is sometimes not helpful: see chapters 9 (Bessel for integer order; but formulae valid for all orders) vs 10 (Bessel of fractional order, meaning $n + \frac{1}{2}$).
- A+S has bugs: 6.1.23: $\ln \Gamma(\overline{z}) = \overline{\ln \Gamma(z)}$ is false: $z = -2.5$.

1.2 Difficulties with OpenMath

- Functions defined by ordinary differential equations:
  need a way of specifying the differential equations and boundary/initial conditions — `odesoln1.ocd`.
- Functions defined by contour integrals:
  we need a way of specifying contours and the restrictions on them — `contour1.ocd`.
- No direct way of specifying the n-th total derivative: this is probably a bug even in MathML conformity.
- Need to specify a restricted range, e.g. only g_2 and g_3 exist as Weierstrass invariants:
  extend the use of STS (but not the syntax).
- Need to specify Cauchy principal value of integrals: an attribute or a new symbol?
  New symbol in calculus2.ocd.
- The 8+3 rule proves restrictive: I want to put the Weierstrass symbols in`weierstrass.ocd`, not `weierstr.ocd`. In fact, this rule is not followed already — see `relation1.ocd` for example.
- Hard to express statements about uniqueness of solutions, or linear independence of solutions.

1.3 `odesoln1.ocd`

```
<OMBIND>
  <OMS name="ODEsolution" cd="odesoln1"/>
  <OMBVAR>
    <OMV name="f"/>
    <OMV name="x"/>
  </OMBVAR>
```

Expression in f and x defining LHS of ODE (=0).

```
  <OMA>
    <OMS name="list" cd="list1"/>
```

List of expressions in f representing initial/*boundary* conditions.

```
  </OMA>
</OMBIND>
```

1.4 `contour1.ocd`

The symbol `path_description` describes a contour. possibly not uniquely.

```
A contour from t to infinity, not crossing the
negative real axis.
<OMOBJ>
```

```
<OMBIND>
  <OMS name="path_description" cd="contour1"/>
  <OMBVAR>
    <OMVAR name="P"/>
  </OMBVAR>
  <OMV name="t"/>
  <OMS name="infinity" cd="nums1"/>
  <OMA>
    <OMS name="not" cd="logic1"/>
    <OMBIND>
      <OMS name="exists" cd="quant1"/>
      <OMBVAR>
        <OMV name="x"/>
      </OMBVAR>
      <OMA>
        <OMS name="and" cd="logic1"/>
        <OMA>
          <OMS name="gt" cd="relation1"/>
          <OMV name="x"/>
          <OMS name="zero" cd="alg1"/>
        </OMA>
        <OMA>
          <OMS name="le" cd="relation1"/>
          <OMA>
            <OMS name="real" cd="complex1"/>
            <OMA>
              <OMV name="P"/>
              <OMV name="x"/>
            </OMA>
          </OMA>
          <OMS name="zero" cd="alg1"/>
        </OMA>
        <OMA>
          <OMS name="eq" cd="relation1"/>
          <OMA>
            <OMS name="imaginary" cd="complex1"/>
            <OMA>
              <OMV name="P"/>
              <OMV name="x"/>
            </OMA>
          </OMA>
          <OMS name="zero" cd="alg1"/>
        </OMA>
      </OMA>
```

```
      </OMBIND>
    </OMA>
  </OMBIND>
</OMOBJ>
```


2 The Real Problem: Analytic Continuation

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t : \qquad x > 0.$$

But, Ei is defined by analytic continuation for all z, continuous modulo a branch cut along the negative real axis.

How do we define this?

- Define a domain for Ei: $\mathbf{C} \setminus \{0\}$.
  But how do we define the branch cut?
- Define a domain of continuity, i.e. $\mathbf{C} \setminus \{\Re(z) < 0; \Im(z) = 0\}$.
  Then how do we say which side the branch cut adheres to?

Suggestions welcome.

3 Bessel Functions

- J_ν and $\mathrm{J}_{-\nu}$ are two independent solutions to Bessel's equation.
- Except when ν is an integer, when

  $$\mathrm{Y}_\nu(z) = \lim_{\mu \to \nu} \frac{\mathrm{J}_\mu(z)\cos(\mu\pi) - \mathrm{J}_{-\mu}(z)}{\sin(\mu\pi)}$$

  is the second solution.
- Unfortunately, the initial conditions at $z = 0$ are not always easy to state.
- $\mathrm{ber}_\nu(x) + i\,\mathrm{bei}_\nu(x) = \mathrm{J}_\nu(xe^{3\pi i/4})$ — do we need this, or the very many other variants?
- Need a domain expert (DLMF?)!